Math 3235 Probability Theory

1/24/23

$\{(H, H), (H, T)\} = A$

$\{(H, H), (T, H)\} = B$

$A \cap B = \{(H, H)\}$

$\mathbb{P}(\{(H, H)\}) = \frac{1}{4}$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/4}{1/2} = \frac{1}{2}$$

(H, H) (T, H)

(H, T) (T, T)

Probability Space (1)

Chapter 2

Random Variable

Discrete r.v.

Ω all what can happen.

$\Omega = \{H, T\}$

$X : \Omega \to \mathbb{R}$

$X(H) = 1 \qquad X(T) = 0$

N coin flips

$\Omega = \{0,1\}^N = \{\sigma = (\sigma_1 \dots \sigma_N)$

with $\sigma_i \in \{H, T\}\}$

$X_i = \begin{cases} 1 & \text{if the i-th flip is } H \\ 0 & \text{" " " " " } T \end{cases}$

Y = # of H is my sequence of flips

T = The position of The first H.

These are functions from

$\Omega \to \mathbb{R}$.

Random variable.

Y number of H

$\mathbb{P}(Y=0) = \mathbb{P}$(set of all sequences That contain no H)

$= 2^{-N}$ (fair coin)

If we have a $(\Omega, \mathcal{F}, \mathbb{P})$

we call a discrete r.v. a function X from Ω into $\mathbb{R}$

1) $X(\Omega)$ is countable.

2) if x is a possible value

of X $\quad (x \in \operatorname{Im}(X))$

we want to define

$$\mathbb{P}(X = x) =$$

$$\mathbb{P}(\{\omega \mid X(\omega) = x\})$$

$$A = \{\omega \mid X(\omega) = x\}$$

$A \subset \Omega$ $\quad$ A must be an event, $\quad A \in \mathcal{F}$

2) The counter image of $x \in \operatorname{Im}(X)$ is an event.

If X is a discrete r.v.
The probability mass function of X
p.m.f.

$$p_X(x) = \mathbb{P}(X = x).$$

Coin flip.

X has only 2 possible values

$$\mathrm{Im}(X) = \{a, b\}$$

Bernoulli situation or r.v.

$$p(a) = \mathbb{P}(X = a) = p$$

$$p(b) = \mathbb{P}(X = b) = 1 - \mathbb{P}(X = a)$$

Y $\quad \mathrm{Im}(Y) = \{0, 1\}$

$$\mathbb{P}(Y = 0) = p$$

$$X = bY + a(1 - Y)$$

$$\mathbb{P}(X = a) = \mathbb{P}(Y = 0)$$

$$\mathbb{P}(X = b) = \mathbb{P}(Y = 1)$$

If you have a r.v. That Takes only Two values, you can always write it as a function of a r.v. with values $\{0, 1\}$.

A r.v. X with possible value $\{0, 1\}$ and $\mathbb{P}(X = 1) = p$ is called a Bernoulli r.v. with par p.

X r.v. The position of The first H.

$$\mathrm{Im}(X) = \{1, 2, \dots \infty\}$$

probability of H is p

$\mathbb{P}(X = 1) = \mathbb{P}(\text{first flip is } H) = p$

$\mathbb{P}(X = 2) = \mathbb{P}(1st = T \ \& \ 2nd = H) =$

$= \mathbb{P}(1st = T)\, \mathbb{P}(2nd = H \mid 1st = T)$

$= \mathbb{P}(1st = T)\, \mathbb{P}(2nd = H) =$

$= (1 - p)\, p$

$\mathbb{P}(X = 3) = (1 - p)^2\, p$

$p_X(x) = (1 - p)^{x-1}\, p$

$\Omega = \{ \underline{\sigma} \mid \underline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_N, \ldots) \}$

$\sigma_i \in \{H, T\}$

$A = \{ \underline{\sigma} \mid \sigma_1 = H \}$

$B = \{ \underline{\sigma} \mid \sigma_i = H, \sigma_j = T, \sigma_k = H \}$

if A is defined by fixing a finite number of outcomes Than A is an event.

$\mathcal{F}$ The smallest subset of The power set of Ω That contains The A defined above and satisfy the axioms.

$$P_X(x) = (1-p)^{x-1}\,p$$

$$\sum_{x=1}^{\infty} P_X(x) = 1$$

(often I can define $P_X(x) = 0$ $x = 0$ or integer < 0).

$$\sum_{x=1}^{\infty} p\,(1-p)^{x-1} = 1$$

if $|p|<1$

$$\sum_{n=0}^{\infty} p^{n} = \frac{1}{1-p}$$

$$\sum_{x=1}^{\infty} P_X(x) = p \sum_{x=1}^{\infty} (1-p)^{x-1} =$$

$$= p \sum_{y=0}^{\infty} (1-p)^{y} = p \frac{1}{1-(1-p)} = 1$$

X with p.m.f

$$P_X(x) = p(1-p)^{x-1}$$

is called a Geometric r.v. with par p.

I fixed number of flips. N

X be the number of H in the outcome. Prob of a H is p and the flips are indep.

$$\mathbb{P}(X=0) = (1-p)^N$$

$$\mathbb{P}(X=1) =$$

$N=3$

HTT $\quad p(1-p)^2$
THT
TTH

$$\mathbb{P}(X=1) = N\,p(1-p)^{N-1}$$

$$\mathbb{P}(X=2) = \binom{N}{2} p^2 (1-p)^{N-2}$$

$$\binom{N}{m} = \frac{N!}{(N-m)!\,m!}$$

N coins

$\circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \ldots \quad \circ$

m of them are H

$m = 1 \qquad N$ possibilities

$m = 2 \qquad \dfrac{N(N-1)}{2}$

m $$\frac{N(N-1)(N-2)\cdots(N-m+1)}{m!}$$

$$= \frac{N!}{(N-m)!\,m!} = \binom{N}{m}$$

$$P_X(x) = \binom{N}{m} p^m (1-p)^{N-m}$$

Binomial p, N.